Chapter 1. ARITHMETIC

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We begin by introducing the following nested family of sets. Let W_n (with $n \in \mathrm{N}^*$) be the set of all decimal fractions, such that there are not more than n digits in the integer part and n digits in the decimal part of the fraction. So, if $x \in W_n$, then $x = a_0.a_1...a_n$ with a_0 being the integer part and $a_1,...,a_n \in \{0,1,2,3,4,5,6,7,8,9\}$. Visually, x looks like $\underbrace{\_\ \ ...\ \ \_}_{n}.\underbrace{\_\ \ ...\ \ \_}_{n}$. If $n < m$, then W_n naturally embeds into W_m by placing 0's in the $n+1^{\text{st}}$ through m^{th} decimal place. If we call the embedding $\varphi_{n,m} : W_n \to W_m$, then for example, let $2.34 \in W_2$ and then $\varphi_{2,4}(2.34) = 2.3400 \in W_4$. Similarly, W_m projects onto W_n by cutting off the superfluous digits on the left and on the right of the decimal point. Let $\phi_{m,n} : W_m \to W_n$ be the projection, then, for example, if $45.4301 \in W_4$, then $\phi_{4,2}(45.4301) = 45.43 \in W_2$.

Let V_n be the set W_n endowed with the following arithmetic structure. If $a_0.a_1...a_n$, $b_0.b_1...b_n \in W_n$, then:

1. $a_0.a_1...a_n = a_0 + 0.a_1 + 0.0a_2 + ... + 0.\underbrace{0...0}_{n-1}a_n$ by definition (this sum already exists in W_n from the point of view of the W_n-observer);
2. $-a_0.a_1...a_n = -a_0 - 0.a_1 - 0.0a_2 - ... - 0.\underbrace{0...0}_{n-1}a_n$;
3. Ordering is defined: if $a_0.a_1...a_n \geq 0$ and $b_0.b_1...b_n \geq 0$, then $a_0.a_1...a_n > b_0.b_1...b_n$ iff $\exists$ $m \in \{1,...,n\} \ni a_i = b_i$ for $0 \leq i \leq m-1$, but $a_m > b_m$; if $a_0.a_1...a_n < 0$ and $b_0.b_1...b_n < 0$, then $a_0.a_1...a_n \geq b_0.b_1...b_n$ iff $-b_0.b_1...b_n \geq -a_0.a_1...a_n$; if $a_0.a_1...a_n \geq 0$ and $b_0.b_1...b_n \leq 0$, then $a_0.a_1...a_n \geq b_0.b_1...b_n$;
4. Addition, $+_n$, is defined: if $a_0.a_1...a_n \geq 0$ and $b_0.b_1...b_n \geq 0$, then

$$a_0.a_1...a_n +_n b_0.b_1...b_n = \left(...\left(\left(\left(a_0 + b_0\right) + 0.a_1 + 0.b_1\right) + 0.0a_2 + 0.0b_2\right) + ... + 0.\underbrace{0...0}_{n-1}a_n + 0.\underbrace{0...0}_{n-1}b_n \right)$$

iff the contents of any parenthesis are in W_n. For example, let $46, 55 \in W_2$, then

$46 +_2 55 = (46+55) = 101 \notin W_2$, hence addition is not defined for these particular elements. If

$a_0.a_1...a_n \le 0$ and $b_0.b_1...b_n \le 0$, then $a_0.a_1...a_n +_n b_0.b_1...b_n = -((-a_0.a_1...a_n) +_n (-b_0.b_1...b_n))$. If

$a_0.a_1...a_n \ge 0$ and $b_0.b_1...b_n \le 0$ and if $a_0.a_1...a_n \ge |b_0.b_1...b_n|$, then $a_0.a_1...a_n +_n b_0.b_1...b_n = c_0.c_1...c_n$,

where the c_i's are computed via the following algorithm. Let $d_k = a_k - |b_k|$, $d_n = a_n - |b_n|$, and

$g_n = d_n$, with the differences taken in the usual sense in W_n. Let $c_n = \begin{cases} g_n \Leftrightarrow g_n \ge 0 \ (i) \\ 10 + g_n \Leftrightarrow g_n < 0 \ (ii) \end{cases}$.

In case (i), let $g_{n-1} = d_{n-1}$ and in case (ii) let $g_{n-1} = d_{n-1} - 1$. Then

$c_{n-1} = \begin{cases} g_{n-1} \Leftrightarrow g_{n-1} \ge 0 \ (i) \\ 10 + g_{n-1} \Leftrightarrow g_{n-1} < 0 \ (ii) \end{cases}$ and again, in case (i), let $g_{n-2} = d_{n-2}$ and in case (ii) let

$g_{n-2} = d_{n-2} - 1$. Proceeding in this way, we get $c_k = \begin{cases} g_k \Leftrightarrow g_k \ge 0 \ (i) \\ 10 + g_k \Leftrightarrow g_k < 0 \ (ii) \end{cases}$ and in case (i), let

$g_{k-1} = d_{k-1}$ and in case (ii) let $g_{k-1} = d_{k-1} - 1$ for $k = 1, 2, ..., n$. Finally, $c_0 = g_0$. Now, since

$a_0.a_1...a_n \ge |b_0.b_1...b_n|$, $\exists\, m \ni g_m \ge 0$, $g_{m-1} \ge 0, \ldots, g_0 > 0$. Next, if $a_0.a_1...a_n \le |b_0.b_1...b_n|$, then

$a_0.a_1...a_n +_n b_0.b_1...b_n = -(-a_0.a_1...a_n +_n (-b_0.b_1...b_n))$. Here are some examples of the above

algorithms. Let 3.14, -5.34, -7.18 and $14.71 \in W_2$, then

$3.14 +_n 14.71 = (((3+14)+0.1+0.7)+0.04+0.01) = 17.85$;

$-5.34 +_2 (-7.18) = -((-(-5.34)) +_2 (-(-7.18))) = -(5.34 +_2 7.18) = -12.52$;

$14.71 +_n (-5.34) = c_0.c_1c_2$, where the c_i's are computed as follows: $d_0 = 14 - |-5| = 9$,

$d_1 = 7 - |-3| = 4$, $d_2 = 1 - |-4| = -3$, and $g_2 = d_2 = -3$. Therefore, $c_2 = 10 + (-3) = 7$ and hence

$g_1 = d_1 - 1 = 3$, so $c_1 = g_1 = 3$ and $c_0 = g_0 = d_0 = 9$. Finally,

$3.14 +_2 (-5.34) = -(-3.14 +_2 (-(-5.34))) = -(5.34 +_2 (-3.14)) = -2.2$ by previous algorithm.

5. Multiplication, $\times_n$, is defined: if $a_0.a_1...a_n \geq 0$ and $b_0.b_1...b_n \geq 0$, then

$$a_0.a_1...a_n \times_n b_0.b_1...b_n = \sum_{k=0}^{n}\sum_{m=0}^{n-k} 0.\underbrace{0...0}_{k-1}a_k \cdot 0.\underbrace{0...0}_{m-1}b_m \text{ where } 0.\underbrace{0...0}_{-1}a_0 = a_0 \text{ and } 0.\underbrace{0...0}_{-1}b_0 = b_0 \text{, i.e.}$$

$$a_0.a_1...a_n \times_n b_0.b_1...b_n = (...((((a_0 \cdot b_0) + a_0 \cdot 0.b_1) + a_0 \cdot 0.0b_2) + ...) + a_0 \cdot 0.\underbrace{0...0}_{n-1}b_n) +$$
$$+0.a_1 \cdot b_0) + 0.a_1 \cdot 0.b_1) + 0.a_1 \cdot 0.0b_1) + ...) + 0.a_1 \cdot 0.\underbrace{0...0}_{n-2}b_{n-1}) + 0.0a_2 \cdot b_0) + 0.0a_2 \cdot 0.b_1) +$$
$$+0.0a_2 \cdot 0.0b_1) + ...) + 0.0a_2 \cdot 0.\underbrace{0...0}_{n-3}b_{n-2}) + ...) + 0.\underbrace{0...0}_{k-1}a_k \cdot b_0) + 0.\underbrace{0...0}_{k-1}a_k \cdot 0.b_1) +$$
$$+0.\underbrace{0...0}_{k-1}a_k \cdot 0.0b_2) + ...) + 0.\underbrace{0...0}_{k-1}a_k \cdot 0.\underbrace{0...0}_{n-k-1}b_{n-k}) + ...) + 0.\underbrace{0...0}_{n-1}a_n \cdot b_0)$$

iff the contents of any parenthesis are in W_n. If $a_0.a_1...a_n \leq 0$ and $b_0.b_1...b_n \leq 0$, then

$a_0.a_1...a_n \times_n b_0.b_1...b_n = |a_0.a_1...a_n| \times_n |b_0.b_1...b_n|$. If $a_0.a_1...a_n \geq 0$ and $b_0.b_1...b_n \leq 0$, then

$a_0.a_1...a_n \times_n b_0.b_1...b_n = -|a_0.a_1...a_n| \times_n |b_0.b_1...b_n|$. For example, let 0.51, 4.48 and $-4.48 \in W_2$, then

$0.51 \times_2 4.48 = (0.5 + 0.01) \cdot (4 + 0.4 + 0.08) = 2 + 0.2 + 0.04 = 2.24$, and

$0.51 \times_2 (-4.48) = -(0.5 + 0.01) \cdot |(-4 - 0.4 - 0.08)| = -(0.5 + 0.01) \cdot (4 + 0.4 + 0.08) = -2.24$. An

example for when multiplication is not defined is the following: let 90, $1.5 \in W_2$, then

$90 \times_2 1.5 = 90 \cdot (1 + 0.5) = ((90 \cdot 1) + 90 \cdot 0.5) = 90 + 45 = 135 \notin W_2$.

Note, if $a = a_0.a_1...a_n \in W_n$ and $b = b_0.b_1...b_n \in W_n$, then $a, b \in W_m$ with $m > n$ by above and the operation of addition in W_n and W_m are equivalent, i.e. $a +_n b = a +_m b$, however, multiplication is not equivalent, $a \times_n b \neq a \times_m b$. For example, if $a = 2.14$, $b = 0.17 \in W_2 \subset W_4$, then

$a +_2 b = a +_4 b = 2.31$ and $a \times_2 b = 0.35 \neq a \times_4 b = 0.3638$.

Now, let us look at some main properties of V_n. So, consider W_n and V_n, as defined above. The following conditions are satisfied:

1. $\forall x, y \in W_{n-1}$ with $n \geq 2$ we have $\varphi_{n-1,n}x +_n \varphi_{n-1,n}y \in W_n$, $\varphi_{n-1,n}y +_n \varphi_{n-1,n}x \in W_n$ and

   $\varphi_{n-1,n}x +_n \varphi_{n-1,n}y = \varphi_{n-1,n}y +_n \varphi_{n-1,n}x$;

2. $\forall x, y \in W_{Ent[0.5n]}$ with $n \geq 2$ we have $\varphi_{Ent[0.5n],n} x \times_n \varphi_{Ent[0.5n],n} y \in W_n$,

$\varphi_{Ent[0.5n],n} y \times_n \varphi_{Ent[0.5n],n} x \in W_n$ and $\varphi_{Ent[0.5n],n} x \times_n \varphi_{Ent[0.5n],n} y = \varphi_{Ent[0.5n],n} y \times_n \varphi_{Ent[0.5n],n} x$;

3. If $m > n$, and $x \in W_n$, $y \in W_m$, then $\phi_{m+1+k,m+1}\left(\varphi_{n,m+1+k} x +_{m+1+k} \varphi_{m,m+1+k} y\right) = \varphi_{n,m+1} x +_{m+1} \varphi_{n,m+1} y$

and $\phi_{m+n+k,m+n}\left(\varphi_{n,m+n+k} x \times_{m+n+k} \varphi_{m,m+n+k} y\right) = \varphi_{n,m+n} x \times_{m+n} \varphi_{n,m+n} y$ for $k = 1, 2, \ldots$.

4. $\forall x, y, z \in W_{n-2}$ with $n \geq 3$ we have $\varphi_{n-2,n} x +_n \varphi_{n-2,n} y \in W_n$, $\left(\varphi_{n-2,n} x +_n \varphi_{n-2,n} y\right) +_n \varphi_{n-2,n} y \in W_n$,

$\varphi_{n-2,n} y +_n \varphi_{n-2,n} z \in W_n$, $\varphi_{n-2,n} x +_n \left(\varphi_{n-2,n} y +_n \varphi_{n-2,n} y\right) \in W_n$,

and $\left(\varphi_{n-2,n} x +_n \varphi_{n-2,n} y\right) +_n \varphi_{n-2,n} y = \varphi_{n-2,n} x +_n \left(\varphi_{n-2,n} y +_n \varphi_{n-2,n} y\right)$;

5. $\forall x, y, z \in W_{Ent[0.25n]}$ with $n \geq 4$ we have $\varphi_{Ent[0.25n],n} x \times_n \varphi_{Ent[0.25n],n} y \in W_n$,

$\left(\varphi_{Ent[0.25n],n} x \times_n \varphi_{Ent[0.25n],n} y\right) \times_n \varphi_{Ent[0.25n],n} y \in W_n$, $\varphi_{Ent[0.25n],n} y \times_n \varphi_{Ent[0.25n],n} z \in W_n$,

$\varphi_{Ent[0.25n],n} x \times_n \left(\varphi_{Ent[0.25n],n} y \times_n \varphi_{Ent[0.25n],n} z\right) \in W_n$ and

$\left(\varphi_{Ent[0.25n],n} x \times_n \varphi_{Ent[0.25n],n} y\right) \times_n \varphi_{Ent[0.25n],n} z = \varphi_{Ent[0.25n],n} x \times_n \left(\varphi_{Ent[0.25n],n} y \times_n \varphi_{Ent[0.25n],n} z\right)$;

6. $\forall x, y, z \in W_{Ent[0.3n]}$ with $n \geq 3$ we have $\varphi_{Ent[0.3n],n} y +_n \varphi_{Ent[0.3n],n} z \in W_n$,

$\varphi_{Ent[0.3n],n} x \times_n \left(\varphi_{Ent[0.3n],n} y +_n \varphi_{Ent[0.3n],n} z\right) \in W_n$, $\varphi_{Ent[0.3n],n} x \times_n \varphi_{Ent[0.3n],n} y \in W_n$,

$\varphi_{Ent[0.3n],n} x \times_n \varphi_{Ent[0.3n],n} z \in W_n$, and finally,

$\varphi_{Ent[0.3n],n} x \times_n \left(\varphi_{Ent[0.3n],n} y +_n \varphi_{Ent[0.3n],n} z\right) = \varphi_{Ent[0.3n],n} x \times_n \varphi_{Ent[0.3n],n} y +_n \varphi_{Ent[0.3n],n} x \times_n \varphi_{Ent[0.3n],n} z$;

7. If $m > n$ then $\forall x, y, z \in W_{Ent[0.3n]}$ with $n \geq 3$ we have $\phi_{m,n}\left(\varphi_{Ent[0.3n],m} x \times_m \varphi_{Ent[0.3n],m} y\right)$,

$\phi_{m,n}\left(\varphi_{Ent[0.3n],m} x \times_m \left(\varphi_{Ent[0.3n],m} y \times_m \varphi_{Ent[0.3n],m} z\right)\right)$, $\phi_{m,n}\left(\varphi_{Ent[0.3n],m} y \times_m \varphi_{Ent[0.3n],m} z\right)$,

$\phi_{m,n}\left(\left(\varphi_{Ent[0.3n],m} x \times_m \varphi_{Ent[0.3n],m} y\right) \times_m \varphi_{Ent[0.3n],m} z\right) \in W_n$ and hence

$\phi_{m,n}\left(\varphi_{Ent[0.3n],m} x \times_m \left(\varphi_{Ent[0.3n],m} y \times_m \varphi_{Ent[0.3n],m} z\right)\right) = \phi_{m,n}\left(\left(\varphi_{Ent[0.3n],m} x \times_m \varphi_{Ent[0.3n],m} y\right) \times_m \varphi_{Ent[0.3n],m} z\right)$.

8. $\exists\ 0 = 0.\underbrace{0 \ldots 0}_{n} \in W_n \ni a +_n 0 = 0 +_n a = a \quad \forall a \in W_n$;

9. $\exists\ 1 = 1.\underbrace{0...0}_{n} \in W_n \ni a \times_n 1 = 1 \times_n a = a \quad \forall a \in W_n$;

10. $\forall x \in W_n \ \exists\ -x \in W_n \ni x +_n (-x) = -x +_n x = 0$;

Note, that for $n \geq 4$, $n-1 > n-2 \geq Ent[0.3n] \geq Ent[0.25n]$, hence

$W_n \supset W_{n-1} \supset W_{n-2} \supset W_{Ent[0.3n]} \supset W_{Ent[0.25n]}$.

11. If $x, y \in W_n$ and $x +_n y \in W_n$ then $y +_n x \in W_n$ and $x +_n y = y +_n x$;

12. If $x, y \in W_n$ and $x \times_n y \in W_n$ then $y \times_n x \in W_n$ and $x \times_n y = y \times_n x$;

13. If $x, y, z \in W_n$ and $x +_n y \in W_n$, $(x +_n y) +_n z \in W_n$, $y +_n z \in W_n$, and $x +_n (y +_n z) \in W_n$ then

$(x +_n y) +_n z = x +_n (y +_n z)$.

Here we provide some basic examples to illustrate what might happen whenever conditions 1-13 above are violated.

1. Additive associativity fails: $(x +_n y) +_n z \neq x +_n (y +_n z)$, e.g. let $10,\ 95,\ -35 \in W_2$, then

$10 +_2 95 \notin W_2$, hence $(10 +_2 95) +_2 (-35) \notin W_2$, but $10 + ({}_2 95 +_2 (-35)) = 70 \in W_2$;

2. Multiplicative associativity fails: $(x \times_n y) \times_n z \neq x \times_n (y \times_n z)$, e.g. let 50.12, 0.85, and

$0.61 \in W_2$, then $50.12 \times_2 0.85 = (50 + 0.1 + 0.02) \cdot (0.8 + 0.05) = 40 + 2.5 + 0.08 = 42.58$, and

$(50.12 \times_2 0.85) \times_2 0.61 = (42 + 0.5 + 0.08) \cdot (0.6 + 0.01) = 25.2 + 0.42 + 0.3 = 25.65$, whereas

$0.85 \times_2 0.61 = (0.8 + 0.05) \cdot (0.6 + 0.01) = 0.48$ and

$50.12 \times_2 (0.85 \times_2 0.61) = (50 + 0.1 + 0.02) \cdot (0.4 + 0.08) = 20 + 4 + 0.04 = 24.04$;

3. Distributivity fails: $x \times_n (y +_n z) \neq x \times_n y +_n x \times_n z$, e.g. let $1.81,\ 0.74,\ 0.53 \in W_2$, then

$0.74 +_2 0.53 = 1.27$ and

$1.81 \times_2 (0.74 +_2 0.53) = (1 + 0.8 + 0.01) \cdot (1 + 0.2 + 0.07) = 1 + 0.2 + 0.07 + 0.8 + 0.16 + 0.01 = 2.24$,

whereas $1.81 \times_2 0.74 = (1 + 0.8 + 0.01) \cdot (0.7 + 0.04) = 0.7 + 0.04 + 0.56 = 1.3$ and

$1.81\times_2 0.53=(1+0.8+0.01)\cdot(0.5+0.03)=0.5+0.03+0.4=0.93$, so that

$1.81\times_2 0.74+_2 1.81\times_2 0.53=2.23$;

4. Lack of the distribution law leads to the following results:

4a. The famous multiplication identities are invalid, e.g. $(x+y)^2 \neq x^2+2(xy)+y^2$, since if we let 0.07 and $0.08\in W_2$, then $0.07+_2 0.08=0.15\in W_2$, and

$0.15\times_2 0.15=(0.1+0.05)\cdot(0.1+0.05)=0.01$, but $0.07^2=0$, $0.08^2=0$, and

$2\times_2 0.07\times_2 0.08=2\times_2 0=0$, so that the RHS is 0, while the LHS is 0.01.

4b. The statement "$x\mid y$ and $x\mid z \Rightarrow x\mid(y+z)$" is false. Here $x\mid y \Leftrightarrow \exists r \ni x\times_n r=y$. Assume that $x\mid y$ and $x\mid z$, what we want to show is equivalent to showing that $y+z\neq x\times_n(r_1+_n r_2)$ for some x,y,z,r_1 and r_2. Let $x=0.17$, $r_1=0.85$, $r_1=0.63$,

$y=0.17\times_2 0.85=0.(0.1+0.07)\cdot(0.8+0.05)=0.08$ and

$z=0.17\times_2 0.63=0.(0.1+0.07)\cdot(0.6+0.03)=0.06$. Then $y+z=0.14$, but $r_1+r_2=1.48$ and

$0.17\times_2 1.48=(0.1+0.07)\cdot(1+0.4+0.08)=0.1+0.04+0.07=0.21$. In fact, $x\nmid y+z$. This is because if we let $0.17\times_2 0.9=(0.1+0.07)\cdot 0.9=0.09<0.14$ and

$0.17\times_2 0.99=(0.1+0.07)\cdot(0.9+0.09)=0.09<0.14$, but $0.17\times_2 1=0.17>0.14$.

5. Multiplicative inverses do not necessarily exist, or if they do, they are not necessarily unique in W_n. Here are some examples: let $2\in W_n$, then $0.5\in W_2$ is the unique inverse of 2 for any W_n. On the other hand, 3 will not have an inverse in any W_n. Now, let $2^{-1}=0.5$, then $(0.5)^{-1}$ is actually the following set $\{2,2.01,2.02,2.03,2.04,2.05,2.06,2.07,2.08,2.09\}\in W_2$. Therefore, $(2^{-1})^{-1}$ is not necessarily 2, hence all we can claim is that if x^{-1} exists, then $x\in\left\{(x^{-1})^{-1}\right\}$.

Further, if an inverse of an element exists in W_n, it does not necessarily exist in W_m for $m\neq n$, independent of the order of m and n, e.g. if $0.91\in W_2$, then

$(0.91)^{-1} = \{1.1, 1.11, 1.12, 1.13, 1.14, 1.15, 1.16, 1.17, 1.18, 1.19\} \in W_2$, but $(0.91)^{-1} \notin W_4$, on the other hand, $16^{-1} = 0.0625 \in W_4$, but $16^{-1} \notin W_2$.

6. Square roots do not necessarily exist. Some examples are, if $4 \in W_n$, then $\sqrt{4} = 2$ for any n and $\sqrt{3}$ does not exist in any W_n. To show that, consider

$1.75 \times_2 1.75 = (1+0.7+0.05)\cdot(1+0.7+0.05) = 1+0.7+0.05+0.7+0.49+0.05 = 2.99$ and

$1.76 \times_2 1.76 = (1+0.7+0.06)\cdot(1+0.7+0.06) = 1+0.7+0.06+0.7+0.49+0.06 = 3.01$.

Further, if a square root of an element exists in W_n, it does not necessarily exist in W_m for $m \neq n$, independent of the order of m and n, e.g. $\sqrt{2} = 1.42 \in W_2$, since

$1.42 \times_2 1.42 = (1+0.4+0.02)\cdot(1+0.4+0.02) = 1+0.4+0.02+0.4+0.16+0.02 = 2$, but $\sqrt{2} \notin V_4$, since

$1.4143 \times_4 1.4143 = (1+0.4+0.01+0.004+0.0003)\cdot(1+0.4+0.01+0.004+0.0003) = 1.9999$

and $1.4144 \times_4 1.4144 = 2.0001$. Also, $\sqrt{1.01} = 1.005 \in W_4$, since

$1.005 \times_4 1.005 = (1+0.005)\cdot(1+0.005) = 1+0.005+0.005 = 1.01$, but $\sqrt{1.01} \notin W_2$, since $1 \times_2 1 = 1$ and $1.01 \times_2 1.01 = (1+0.01)\cdot(1+0.01) = 1+0.01+0.01 = 1.02$.

Next, some basic theorems can be stated for W_n:

1. Any W_n has zero divisors: $0.\underbrace{0...0}_{n-1}1 \times_n 0.\underbrace{0...0}_{n-1}1 = 0$;
2. If $p \in W_n$ with $p \neq 2,5$ a prime in the usual sense, then $p^{-1} \notin W_n$ for any W_n;
3. $\forall x, y \in W_n$ with $x, y \geq 0$, $x - y \in W_n$.
4. In view of example 6 above, we have that if $x, y, t, u \in W_n$ and $x \geq t \geq 0$ and $y \geq u \geq 0$, then $x \times_n y \geq t \times_n u$
5. If given $a \in W_n$ such that there is a unique $a^{-1} \in W_n$, then $|a| \geq 1$;

6. If $|a| < 1$ and a^{-1} exists, then $\left|\{a^{-1}\}\right| > 1$;

7. If $\left|\{a^{-1}\}\right| > 1$, then $|a| < 1$.

In general, the sets V_n and V_m are not dependent on each other for $n \neq m$. To illustrate this situation, let us look at the following examples.

1. Multiplicative associativity and distributivity in V_m (if $m > n$) remain after projecting the elements onto V_n. Suppose $x = 2.0034$, $y = 3.0001$, and $z = 4.0027 \in W_4$, then $x \times_4 y = 6.0104$, $(x \times_4 y) \times_4 z = 24.0578$, $y \times_4 z = 12.0085$, and $x \times_4 (y \times_4 z) = 24.0578$, so that associativity holds in V_4. Next, $(x +_4 y) \times_4 z = 20.0275$, $x \times_4 z = 8.019$, and $y \times_4 z = 12.0085$ so that $x \times_4 z +_4 y \times_4 z = 20.0275$ and distributivity holds in V_4. Now, $\phi_{4,2}(x) = 2$, $\phi_{4,2}(y) = 3$, and $\phi_{4,2}(z) = 4$ so that associativity and distributivity obviously hold in V_2.

2. Multiplicative associativity does not hold in both V_n and V_m, but distributivity holds in either one (after appropriate projections). Suppose $x = 0.2034$, $y = 0.3001$, and $z = 0.4027 \in W_4$, then $x \times_4 y = 0.0609$, $(x \times_4 y) \times_4 z = 0.024$, $y \times_4 z = 0.1206$, and $x \times_4 (y \times_4 z) = 0.0243$, so that associativity fails in V_4. Next, $\phi_{4,2}(x) = 0.2$, $\phi_{4,2}(y) = 0.3$, and $\phi_{4,2}(z) = 0.4$ so that $0 = (\phi_{4,2}(x) \times_2 \phi_{4,2}(y)) \times_2 \phi_{4,2}(z) \neq \phi_{4,2}(x) \times_2 (\phi_{4,2}(y) \times_2 \phi_{4,2}(z)) = 0.02$ and hence associativity fails in V_2. Now, $x +_4 y = 0.5035$, $(x +_4 y) \times_4 z = 0.2022$, $x \times_4 z = 0.0816$, and $y \times_4 z = 0.1206$, so that $x \times_4 z +_4 y \times_4 z = 0.2022$, so distributivity holds in V_4. The fact that distributiviy holds in V_2 is obvious.

3. Multiplicative associativity fails in V_m, but holds in V_n, and distributivity holds in both V_m and V_n (after appropriate projections). Suppose $x = 0.2034$, $y = 0.3001$, and $z = 4.0027 \in W_4$, then $x \times_4 y = 0.0609$, $(x \times_4 y) \times_4 z = 0.2436$, $y \times_4 z = 1.201$, and $x \times_4 (y \times_4 z) = 0.2442$, so that associativity fails in V_4. Now, $\phi_{4,2}(x) = 0.2$, $\phi_{4,2}(y) = 0.3$, and $\phi_{4,2}(z) = 4$ so that it is clear that associativity holds in V_2. Next, $x +_4 y = 0.5035$, $(x +_4 y) \times_4 z = 2.015$, $x \times_4 z = 0.814$, and $y \times_4 z = 1.201$, so that $x \times_4 z +_4 y \times_4 z = 2.015$, so distributivity holds in V_4 and clearly it holds in V_2.

4. Distributivity fails in V_m, but (after projecting or embedding) holds in V_n. Suppose $x = 0.2079$, $y = 0.3084$, and $z = 0.4027 \in W_4$, then $x +_4 y = 0.5163$, $(x +_4 y) \times_4 z = 0.2074$, $x \times_4 z = 0.0832$, and $y \times_4 z = 0.1238$, so that $x \times_4 z +_4 y \times_4 z = 0.207$, so distributivity fails in V_4. Now, $\phi_{4,2}(x) = 0.2$, $\phi_{4,2}(y) = 0.3$, and $\phi_{4,2}(z) = 0.4$ so that it is clear that distributivity holds in V_2. To show that order of m and n does not matter, assume $x = 0.48$, $y = 0.73$, and $z = 4.46 \in W_2$, then $x +_2 y = 1.21$, $(x +_2 y) \times_2 z = 5.38$, $x \times_2 z = 2.08$, and $y \times_2 z = 3.2$, so that $x \times_2 z +_2 y \times_2 z = 5.28$ and hence distributiviy fails in V_2. Now, $\varphi_{2,4}(x) = 0.4800$, $\varphi_{2,4}(y) = 0.7300$, and $\varphi_{2,4}(z) = 4.4600$ so that $\varphi_{2,4}x +_4 \varphi_{2,4}y = 1.2100$, $(\varphi_{2,4}x +_4 \varphi_{2,4}y) \times_4 \varphi_{2,4}z = 5.3966$, $\varphi_{2,4}x \times_4 \varphi_{2,4}z = 2.1408$, and $\varphi_{2,4}y \times_4 \varphi_{2,4}z = 3.2558$, so that $\varphi_{2,4}x \times_4 \varphi_{2,4}z +_4 \varphi_{2,4}y \times_4 \varphi_{2,4}z = 5.3966$ and distributivity holds in V_4.