Chapter 4. **ANALYSIS and TOPOLOGY**

We begin the discussion of analysis with convergent sequences. Here is the definition of a sequence converging in some W_n from a point of view of W_m observer with $m > n$. Given $x^k = x_0^k.x_1^k...x_n^k$ and $a = a_0.a_1...a_n$, with $k \in \mathbb{Z}^+$, $x^k \to a$ (as k increases) iff there exist $k_0 < k_1 < ... < k_n \in W_n$ (integers) such that $x_0^k = a_0$ for all $k \geq k_0$, …, $x_n^k = a_n$ for all $k \geq k_n$. Of course, locally, or for the W_n observer, $x^k \to a$ convergence can be defined in the usual sense (via ε and N). Similarly, given $b = b_0.b_1...b_n$ and $f\left(x^k\right) = f_0^k.f_1^k...f_n^k$, we have $\begin{cases} f\left(x^k\right) \to b \\ x^k \to a \end{cases}$ iff there exist $k_0 < k_1 < ... < k_n \in W_n$ (integers), such that $\begin{cases} x_0^k = a_0 \\ f_0^k = b_0 \end{cases}$ for all $k \geq k_0$, …, $\begin{cases} x_n^k = a_n \\ f_n^k = b_n \end{cases}$ for all $k \geq k_n$. Again, for the W_n observer the usual definitions take place. Now, from the point of view of the W_l observer (with $l > m$), in W_m we have $x^k = x_0^k.x_1^k...x_m^k$, $a = a_0.a_1...a_m$, $b = b_0.b_1...b_m$ and $f\left(x^k\right) = f_0^k.f_1^k...f_m^k$ and therefore $\begin{cases} f\left(x^k\right) \to b \\ x^k \to a \end{cases}$ iff there exist $k_0 < k_1 < ... < k_m \in W_m$ (integers), such that $\begin{cases} x_0^k = a_0 \\ f_0^k = b_0 \end{cases}$ for all $k \geq k_0$, …, $\begin{cases} x_m^k = a_m \\ f_m^k = b_m \end{cases}$ for all $k \geq k_m$. So, the W_m observer sees infinity, while knows that the W_n observer does not. Similarly, the W_l observer sees infinity, while knows that the other two do not. Let us consider how the W_l observer studies the convergence: does $x \to a$ imply that $f\left(x\right) \to b$? The W_l observer sees what happens with convergences in W_n and W_m, i.e. he pushes x towards a and studies what happens with $f\left(x\right)$ (whether it approaches b or not) in the ε -δ sense and also observes how this is seen by the W_n and W_m observers. Clearly, if W_l observer sees that $f\left(x\right) \to b$ in the ε -δ sense as $x \to a$, then there exist

$k_0 < \ldots < k_n < \ldots < k_m < \ldots < k_l \in W_l$ such that $\begin{cases} x_0^k = a_0 \\ f_0^k = b_0 \end{cases}$ for all $k \geq k_0$, …, $\begin{cases} x_n^k = a_n \\ f_n^k = b_n \end{cases}$ for all $k \geq k_n, \ldots, \begin{cases} x_m^k = a_m \\ f_m^k = b_m \end{cases}$ for all $k \geq k_m$, …, $\begin{cases} x_l^k = a_l \\ f_l^k = b_l \end{cases}$ for all $k \geq k_l$ since the W_l observer can take ε and δ such that $\varepsilon < 0.\underbrace{0...0}_{l}1$ and $\delta < 0.\underbrace{0...0}_{l}1$. However, it is not guaranteed that $k_0, k_1, ..., k_n \in W_n$ and $k_n, k_{n+1}, ..., k_m \in W_m$. This means that if $\begin{cases} f\left(x^k\right) \to b \\ x^k \to a \end{cases}$ for the W_l observer, then it might not be true for the other observers. Also, if $\begin{cases} f\left(x^k\right) \not\to b \\ x^k \to a \end{cases}$ for the W_l observer in the ε-δ sense we can have the following cases occur: $\begin{cases} f\left(x^k\right) \to b \\ x^k \to a \end{cases}$ for the other two observers, $\begin{cases} f\left(x^k\right) \to b \\ x^k \to a \end{cases}$ for W_n and $\begin{cases} f\left(x^k\right) \not\to b \\ x^k \to a \end{cases}$ for W_m observers, and finally, $\begin{cases} f\left(x^k\right) \not\to b \\ x^k \to a \end{cases}$ for W_n and $\begin{cases} f\left(x^k\right) \not\to b \\ x^k \to a \end{cases}$ for W_m observers (the divergence results are independent). Hence, the existence of a limit depends on the observer.

Let us now consider the following set of observers W_{n_1},…, $W_{n_{k+1}}$. Now, if we pick a $W_{n_{s+1}}$-observer with s large enough, such that he sees the following sequence: $a^r = \sum_{l=0}^{r} 10^{-2n_{k+1}+l}$ for $r = 0,1,...,2n_{k+1}$ and $a^{2n_{k+1}+r} = a^{2n_{k+1}} + \sum_{l=1}^{r} 10^{-2n_{k+1}-l}$ for $r = 1,2,...$ (in particular, that means that this observer sees not only the entire $W_{n_{k+1}}$, but also the numbers $10^{-2n_{k+1}+l}$; also, we are using this notation to minimize space). Then this sequence converges to 1.1...1...1, i.e. for every W_{n_i} the sequence stabilizes, but much later than it is possible to view even by the $W_{n_{s+1}}$ observer. This means that only some "future generation" of the $W_{n_{s+1}}$ observer will be able to see the complete stabilization of this sequence. The partial stabilization of the sequence, however, is seen by every generation of the $W_{n_{s+1}}$ observer, who thinks that this is the limit, which is not so.

In fact, this kind of convergence gives birth to the concept of time. Suppose we have $x, a, f(x), b \in W_{n_1}$ such that there exist $k_0, k_1, ..., k_{n_1}$ such that $\begin{cases} x_0^k = a_0 \\ f_0^k = b_0 \end{cases}$ for all $k \geq k_0$, …, $\begin{cases} x_{n_1}^k = a_{n_1} \\ f_{n_1}^k = b_{n_1} \end{cases}$ for all $k \geq k_{n_1}$, but $k_0, k_1, ..., k_{n_1} \notin W_{n_1}$ or even $k_0, k_1, ..., k_{n_1} \notin W_{n_2}$, then, philosophically speaking, the W_{n_1} -, and W_{n_2} -observers do not see the fact of stabilization (convergence), but notice it only after some "time". Hence we call this phenomenon Time. Also, what is invalid for one generation can become valid for a union of a few generations (note here, we talk about a collection of these generations, not some single future generation). In general, an algorithm which contains not more than n steps is valid for only a single generation.