1. ARITHMETIC OPERATIONS IN OBSERVER’S MATHEMATICS

We consider a finite well-ordered system of observers, where each observer sees the real numbers as the set of all infinite decimal fractions. The observers are ordered by their level of “depth”, i.e. each observer has a depth number (hence, we have the regular integer ordering), such that an observer with depth \(k\) sees that an observer with depth \(n < k\) sees and deals (to be defined below) not with an infinite set of infinite decimal fractions, but, actually, with a finite set of finite decimal fractions. We call this set \(W_n\), i.e. it is the set of all decimal fractions, such that there are at most \(n\) digits in the integer part and \(n\) digits in the decimal part of the fraction. Visually, an element in \(W_n\) looks like

\[
\overbrace{...}^{n} \overbrace{...}^{n}.
\]

Moreover, an observer with a given depth is unaware (or can only assume the existence) of observers with larger depth values and for his purposes, he deals with “infinity”. These observers are called naive, with the observer with the lowest depth number – the most naive. However, if there is an observer with a higher depth number, he sees that a given observer actually deals with a finite set of finite decimal fractions, and so on. Therefore, if we fix an observer, then this observer sees the sets \(W_{n_1}, \ldots, W_{n_k}\) with \(n_1 < \ldots < n_k\) indicating the depth level, and realizes that the corresponding observers see and deal with infinity. When we talk about observers, we shall always have some fixed observer (called ‘us’) who oversees all others and realizes that they are naive. The “\(W_n\)-observer” is the abbreviation for somebody who deals with \(W_n\) while thinking that he deals with infinity.

We begin by defining sets \(W_n\) which consist of all finite decimal fractions such that there are at most \(n\) digits in the integer part and at most \(n\) digits in the decimal part. That is, the set \(W_n\) contains all elements of the form \(a = a_0.a_1\ldots a_n\) where the integer part can be written as \(a_0 = b_{n-1}\ldots b_0\), where \(b_{n-1}, \ldots, b_0, a_1, \ldots, a_n \in \{0, 1, \ldots, 9\}\). Now, given \(c = c_0.c_1\ldots c_n, d = d_0.d_1\ldots d_n \in W_n\) we endow \(W_n\) with the following arithmetic \((\pm_n, -_n, \times_n, \div_n)\) - from \(W_m\) observer point of view \((m > n)\):

**Definition 1.1. Addition and subtraction**

\[
c \pm_n d = \begin{cases} 
c \pm d, & \text{if } c \pm d \in W_n \\
\text{not defined, if } c \pm d \notin W_n
\end{cases}
\]

where \(c \pm d\) is the standard addition and subtraction, and we write \(((\ldots (f_1 +_n f_2) \ldots) +_n f_N) = \sum_{i=1}^{N} n f_i\) for \(f_1, \ldots, f_N\) iff the contents of any parenthesis are in \(W_n\), \(f_1, \ldots, f_N \in W_n\).

**Definition 1.2. Multiplication**

\[
c \times_n d = \sum_{k=0}^{n} \sum_{m=0}^{n-k} c_k \cdot 0\ldots0 d_m
\]

where \(c, d \geq 0, c_0 \cdot d_0 \in W_n, 0\ldots0 c_k \cdot 0\ldots0 d_m\) is the standard product, and \(k = m = 0\) means that \(0\ldots0 c_k = c_0\) and \(0\ldots0 d_m = d_0\). If either \(c < 0\) or \(d < 0\), then we compute \(|c| \times_n |d|\) and
define \( c \times_n d = \pm |c| \times_n |d| \), where the sign \( \pm \) is defined as usual. Note, if the content of at least one parentheses (in previous formula) is not in \( W_n \), then \( c \times_n d \) is not defined.

**Definition 1.3. Division**

\[
c \div_n d = \begin{cases} r, & \text{if } \exists r \in W_n \quad r \times_n d = c \\ 
\text{not defined, if no such } r \text{ exists} \end{cases}
\]

Let \( n = 2 \), so we are in \( W_2 \). Here are some examples of elements of \( W_2 \): \( 3.14, -99, 0.1 \in W_2 \) and \( 0.115, 123.9, -100000 \notin W_2 \). Now, the examples of arithmetic:

\[
\begin{align*}
2.08 +_2 11.9 & = 13.98 \\
(-2.08) +_2 11.9 & = 9.82 \\
80 +_2 24 & = \text{not defined} \\
21.36 -_2 0.87 & = 20.49 \\
1.36 -_2 16.95 & = -15.59 \\
1.36 -_2 (-99.95) & = \text{not defined} \\
11 \times_2 8 & = 88 \\
(-5) \times_2 19 & = -95 \\
11 \times_2 12 & = \text{not defined} \\
3.41 \times_2 2.64 & = 8.98 \\
3.41 \times_2 (-2.64) & = -8.98 \\
3.41 \times_2 42.64 & = \text{not defined} \\
99.41 \times_2 1.64 & = \text{not defined} \\
0.85 \times_2 0.02 & = 0 \\
80 \div_2 4 & = 20
\end{align*}
\]

- we get 10 different \( r \)'s

\[
1 \div_2 3 = \text{not defined}
\]

(since no \( r \) exists). In case \( p > q \), \( \star \rightarrow \infty \) for \( W_q \)-observer means \( \star \rightarrow 10^q \) for \( W_p \)-observer, and

\[
\star \rightarrow 0 \text{ for } W_q\text{-observer means } \star \rightarrow 10^{-q} \text{ for } W_p\text{-observer.}
\]

Here we provide some basic examples to illustrate what might happen.

1. Additive associativity fails: \((x +_n y) +_n z \neq x +_n (y +_n z)\), e.g. let \( 10, 95, -35 \in W_2 \), then \( 10 +_2 95 \notin W_2 \), hence \((10 +_2 95) +_2 (-35) \notin W_2 \), but \( 10 +_2 (95 +_2 (-35)) = 70 \in W_2 \);

2. Multiplicative associativity fails: \((x \times_n y) \times_n z \neq x \times_n (y \times_n z)\), e.g. let \( 50.12, 0.85, \) and \( 0.61 \in W_2 \), then \( 50.12 \times_2 0.85 = (50 + 0.1 + 0.02) \cdot (0.8 + 0.05) = 40 + 2.5 + 0.08 = 42.58, \) and
(50.12 \times 2 \times 0.85) \times 2 \times 0.61 = (42 + 0.5 + 0.08) \cdot (0.6 + 0.01) = 25.2 + 0.42 + 0.3 = 25.65, \text{ whereas } 0.85 \times 2 \times 0.61 = (0.8 + 0.05) \cdot (0.6 + 0.01) = 0.48 \text{ and } 50.12 \times 2 \times (0.85 \times 2 \times 0.61) = (50 + 0.1 + 0.02) \cdot (0.4 + 0.08) = 20 + 4 + 0.04 = 24.04;

3. Distributivity fails: \( x \times_n (y + z) \neq x \times_n y + x \times_n z \), e.g. let 1.81, 0.74, 0.53 \in W_2, then \( 0.74 + 0.53 = 1.27 \) and \( 1.81 \times_2 (0.74 + 0.53) = (1 + 0.8 + 0.01) \cdot (1 + 0.2 + 0.07) = 2.24, \) whereas \( 1.81 \times_2 0.74 = (1 + 0.8 + 0.01) \cdot (0.7 + 0.04) = 7 + 0.04 = 7.04 \), and \( 1.81 \times_2 0.53 = (1 + 0.8 + 0.01) \cdot (0.5 + 0.03) = 0.5 + 0.03 + 0.4 = 0.93, \) so that \( 1.81 \times_2 0.74 + 1.81 \times_2 0.53 = 2.23; \)

4. Lack of the distribution law leads to the following result:

The statement “\( x | y \) and \( x | z \) \( \Rightarrow \) \( x | (y + z) \)” is false. Here \( x | y \leftrightarrow \exists r : x \times_n r = y \). Assume that \( x | y \) and \( x | z \), on the other hand, 3 will not have an inverse in any \( W_n \). Now, let \( 2^{-1} = 0.5 \), then \((0.5)^{-1}\) is actually the following set \{2, 2.01, 2.02, 2.03, 2.04, 2.05, 2.06, 2.07, 2.08, 2.09\} \( \subseteq W_2 \). Therefore, \((2^{-1})^{-1}\) is not necessarily 2, hence all we can claim is that if \( x^{-1}\) exists, then \( x \in \{(x^{-1})^{-1}\} \). Further, if an inverse of an element exists in \( W_n \), it does not necessarily exist in \( W_m \) for \( m \neq n \), independent of the order of \( m \) and \( n \), e.g. if \( 0.91 \in W_2 \), then \((0.91)^{-1} = \{1.1, 1.11, 1.12, 1.13, 1.14, 1.15, 1.16, 1.17, 1.18, 1.19\} \subseteq W_2 \), but \((0.91)^{-1} \not\in W_4 \), on the other hand, \(16^{-1} = 0.0625 \in W_4 \), but \(16^{-1} \not\in W_2 \).

5. Multiplicative inverses do not necessarily exist, or if they do, they are not necessarily unique in \( W_n \). Here are some examples: let \( 2 \in W_n \), then \( 0.5 \in W_2 \) is the unique inverse of 2 for any \( W_n \). On the other hand, 3 will not have an inverse in any \( W_n \). Now, let \( 2^{-1} = 0.5 \), then \( (0.5)^{-1} \) is actually the following set \{2, 2.01, 2.02, 2.03, 2.04, 2.05, 2.06, 2.07, 2.08, 2.09\} \( \subseteq W_2 \). Therefore, \((2^{-1})^{-1}\) is not necessarily 2, hence all we can claim is that if \( x^{-1}\) exists, then \( x \in \{(x^{-1})^{-1}\} \). Further, if an inverse of an element exists in \( W_n \), it does not necessarily exist in \( W_m \) for \( m \neq n \), independent of the order of \( m \) and \( n \), e.g. if \( 0.91 \in W_2 \), then \((0.91)^{-1} = \{1.1, 1.11, 1.12, 1.13, 1.14, 1.15, 1.16, 1.17, 1.18, 1.19\} \subseteq W_2 \), but \((0.91)^{-1} \not\in W_4 \), on the other hand, \(16^{-1} = 0.0625 \in W_4 \), but \(16^{-1} \not\in W_2 \).

6. Square roots do not necessarily exist. Some examples are, if \( 4 \in W_n \), then \( \sqrt{4} = 2 \) for any \( n \) and \( \sqrt{3} \) does not exist in \( n = 2 \). To show that, consider \( 1.76 \times_2 1.75 = (1 + 0.7 + 0.05) \cdot (1 + 0.7 + 0.05) = 1 + 0.7 + 0.05 + 0.7 + 0.49 + 0.05 = 2.99 \) and

\[
1.76 \times_2 1.76 = (1 + 0.7 + 0.06) \cdot (1 + 0.7 + 0.06) = 1 + 0.7 + 0.06 + 0.7 + 0.49 + 0.06 = 3.01.
\]

Further, if a square root of an element exists in \( W_n \), it does not necessarily exist in \( W_m \) for \( m \neq n \), independent of the order of \( m \) and \( n \), e.g. \( \sqrt{2} = 1.42 \in W_2 \), since \( 1.42 \times_2 1.42 = (1 + 0.4 + 0.02) \cdot (1 + 0.4 + 0.02) = 1 + 0.4 + 0.02 + 0.4 + 0.16 + 0.02 = 2, \) but \( \sqrt{2} \not\in W_4 \), since \( 1.4143 \times_4 1.4143 = (1 + 0.4 + 0.01 + 0.004 + 0.0003) \cdot (1 + 0.4 + 0.01 + 0.004 + 0.0003) = 1.99999 \)

and \( 1.4144 \times_4 1.4144 = 2.0001. \) Also, \( \sqrt{101} = 1.005 \in W_4 \), since \( 1.005 \times_4 1.005 = (1 + 0.005) \cdot (1 + 0.005) = 1 + 0.005 + 0.005 = 1.01, \) but \( \sqrt{101} \not\in W_2 \), since \( 1 \times_2 1 = 1 \) and \( 1.01 \times_2 1.01 = (1 + 0.01) \cdot (1 + 0.01) = 1 + 0.01 + 0.01 = 1.02. \)

Next, some basic theorems can be stated for \( W_n \):
1. Any $W_n$ has zero divisors: $0, 0\ldots01 \times_n 0, 0\ldots01 = 0$;

2. If $p \in W_n$ with $p \neq 2, 5$ a prime in the usual sense, then $p^{-1} \notin W_n$ for any $W_n$;

3. $\forall x, y \in W_n$ with $x, y \geq 0$, $x - y \in W_n$.

4. If $x, y, t, u \in W_n$ and $x \geq t \geq 0$ and $y \geq u \geq 0$ and $x \times_n y \in W_n$, then $t \times_n u \in W_n$ and $x \times_n y \geq t \times_n u$.

5. If given $a \in W_n$ such that there is a unique $a^{-1} \in W_n$, then $|a| \geq 1$;

6. If $|a| < 1$ and $a^{-1}$ exists, then $|\{a^{-1}\}| > 1$;

7. If $|\{a^{-1}\}| > 1$, then $|a| < 1$.

Let’s consider now additional valuable properties of introduced arithmetic.

1. Standard multiplications identities become wrong, for example

$$(x + y)^2 \neq x^2 + 2(xy) + y^2$$

We have

**Theorem 1.4.** $P((a + n b) \times_n (a + n b) = (a \times_n a + n 2 \times_n (a \times_n b)) +_n b \times_n b) < 1$, where $P$ is a probability. We can see a proof below. Let $n = 2$. Then

1. Left side of equality is $(1.32 + 2.43) \times_2 (1.32 + 2.43) = 3.75 \times_2 3.75 = 13.99$, right side consists from two parts. First, $1.32 \times_2 1.32 = 1.73$; second, $2 \times_2 (1.32 \times_2 2.43) = 6.38$, and finally $2.43 \times_2 2.43 = 5.88$. That means $1.73 + 6.38 + 5.88 = 13.99$. Left side equals to right. But now let’s consider the following calculations.

2. Left side of equality is $(1.32 + 2.79) \times_2 (1.32 + 2.79) = 4.11 \times_2 4.12 = 16.89$, right side consists from two parts. First, $1.32 \times_2 1.32 = 1.73$; second, $2 \times_2 (1.32 \times_2 2.79) = 7.28$, and finally $2.79 \times_2 2.79 = 7.65$. That means $1.73 + 7.28 + 7.65 = 16.66$. Left side does not equal to right.

Let’s consider now a random variable

$$\delta_1 = (a + n b) \times_n (a + n b) - ((a \times_n a + n 2 \times_n (a \times_n b)) +_n (b \times_n b))$$

where $a, b \geq 0$, and $\delta_1$, and all elements of right side belong to $W_n$. Let’s $n = 2$. Using direct calculation we can build $F_1(x)$ - distribution function of $\delta_1$, where

$$F_1(x) = P(\delta_1 < x)$$

, $P$ is a probability. Graph of $F_1(x)$ you can see on Fig. 1.
General proof for $W_n$ you can see below. If $a, b$ non-negative integers in $W_n$ and $(a +_n b) \times_n (a +_n b) \in W_n$, then $\delta_1 = 0$. let’s consider now $a = 0.9\ldots 9$ and $b = 0.0\ldots 08$. Then $a +_n b = 1.0\ldots 07$ and $(a +_n b) \times_n (a +_n b) = 1.0\ldots 07 \times_n 1.0\ldots 07 = 1.0\ldots 014$, but $a \times_n a < 1$, $b \times_n b = 0$, and $2 \times_n (a \times_n b) = 0$. I.e. $\delta_1 \neq 0$.

2. We have also the following theorem.

**Theorem 1.5.**

$$P(c \times_n (a +_n b) = c \times_n a +_n c \times_n b) < 1$$

where $P$ is a probability. Below you can see a proof. Let’s $n = 2$. Then

1. Left side of equality is $2 \times_2 (3 +_2 6) = 2 \times_2 9 = 18$, right side consists from two parts. First, $2 \times_2 3 = 6$, then $2 \times_2 6 = 12$ and $6 +_2 12 = 18$ I.e. left side equals to right. But go to next calculations.

2. Left side of equality is $2.41 \times_2 (3.14 +_2 0.58) = 2.41 \times_2 3.72 = 8.95$, right side consists from two parts. First, $2.41 \times_2 3.14 = 7.55$, then $2.41 \times_2 0.58 = 1.36$ и $7.55 +_2 1.36 = 8.91$. I.e. left side does not equal to right.

Let’s consider a random variable

$$\delta_2 = c \times_n (a +_n b) -_n (c \times_n a +_n c \times_n b)$$
where \(a, b, c \geq 0\), and \(\delta_2\) and all elements of right side belong to \(W_n\). Let’s \(n = 2\). Using direct calculations we can build \(F_2(x)\) - distribution function of \(\delta_2\), where \(F_2(x) = P(\delta_2 < x)\), where \(P\) is a probability. Graph of \(F_2(x)\) you can see on Fig. 2.

General proof for \(W_n\) you can see below. If \(a, b, c\) - non-negative integers in \(W_n\) and \(a \times_n (b \times_n c) \in W_n\), then \(\delta_2 = 0\). Let’s consider now \(a = 2\), \(b = 0.9\ldots9\) and \(c = 0.0\ldots01\). Then \(b \times_n c = 0\), \(a \times_n (b \times_n c) = 0\), \(a \times_n b = 1.9\ldots98\), and \((a \times_n b) \times_n c = 0.0\ldots01\). I.e. \(\delta_2 = 0\).

3.

**Theorem 1.6.** Let’s

\[
\delta_3 = c \times_n (a \times_n b) - n (c \times_n a) \times_n b
\]

Then \(0 < P(\delta_3 = 0) < 1\), where \(P\) is a probability.

You can see a proof of this theorem below. Let’s \(n = 2\). Then

1. Left side of this equality is \(2 \times_2 (3 \times_2 6) = 2 \times_2 18 = 36\), right side consists from two parts. First, \(2 \times_2 3 = 6\), then \(6 \times_2 6 = 36\). I.e left side equals to right. But let’s consider the following calculations.

2. Left side is \(2.41 \times_2 3.14 \times_2 0.58 = 2.41 \times_2 1.79 = 4.27\), for right side we get first, \(2.41 \times_2 3.14 = 7.55\), then \(7.55 \times_2 0.58 = 4.31\). And left side does not equal to right. Let’s consider a random variable

\[
\delta_3 = c \times_n (a \times_n b) - n (c \times_n a) \times_n b
\]
where $a, b, c \geq 0$, and $\delta_3$ and all elements of right side belong to $W_n$. If we take $n = 2$, then using direct calculations we can build $F_3(x)$ - distribution function of $\delta_3$, where $F_3(x) = P(\delta_3 < x)$, and $P$ is a probability. Graph of $F_3(x)$ you can see on Fig. 3. General proof for $W_n$ you can see below. If $a, b, c$ are non-negative integers in $W_n$ and $c \times_n (a \times_n b) \in W_n$, then $\delta_3 = 0$. Let’s consider $c = 2$, $a = 0.\underbrace{9\ldots 99}_n$ and $b = 0.\underbrace{0\ldots 01}_n$. Then

$$\delta_3 = 2 \times_n (0.\underbrace{9\ldots 99}_n \times_n 0.\underbrace{0\ldots 01}_n) - \underbrace{n}_n (2 \times_n 0.\underbrace{9\ldots 99}_n) \times_n 0.\underbrace{0\ldots 01}_n = 0 - n 0.\underbrace{0\ldots 01}_n = -0.\underbrace{0\ldots 01}_n \neq 0$$